Nested linear low-gain design for semiglobal stabilization of feedforward systems*

We show that semiglobal stabilization of a large class of feedforward nonlinear systems is achieved by low-gain linear feedback provided that the separation of the gains is sufficient.

Particular situations are identified where the tuning of the gains only requires increasing powers of a single parameter.

A recursive tuning of independent parameters is necessary in the general caseto avoid vanishing regions of attractions.

Keywords: Nonlinear control, semiglobal stabilization, feedforward systems

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1 Introduction

This paper deals with the problem of semiglobal stabilization of nonlinear systems by using linear feedback: given the system

$$\dot{x} = F(x) + G(x)u, \quad F(0) = 0, \quad x \in I\!R^n \tag{1.1}$$

whose Jacobian linearization is controllable, when and how is it possible to tune the gains of a linear controller $u=Kx$ in order to include an arbitrarily large bounded prescribed set in the region of attraction of the equilibrium $x=0$?

Since its original formulation in [1], the above problem has stimulated important contributions under the form of necessary or sufficient structural conditions on the nonlinearities of (1.1) to achieve arbitrarily large regions of attraction.
Regarding the necessary conditions, counterintuitive obstacles to semiglobal stabilization have been discovered in the analysis of the peaking phenomenon [12, 2, 11].
Regarding the sufficient conditions, most results have been derived from the simplification of nonlinear designs which achieve *global* stabilization.
Thus, for *strict-feedback* systems

$$\left\{\begin{array}{rcl} \dot{z} &=& f(z) + g(z)\xi_1 \\ \dot{\xi}_1 &=& \xi_2 + a_1(z,\xi_1) \\ &\vdots& \\ \dot{\xi}_n &=& u + a_n(z,\xi_1,\dots,\xi_n) \end{array}\right. \tag{1.2}$$

which consist of a core system $\dot{z} = f(z)$ controlled through a chain of nonlinear integrators with all the nonlinearities in feedback form, the backstepping methodology provides a recursive construction of a Control Lyapunov Function [9] which can be employed for the design of a globally stabilizing control law.
A similar recursive approach shows that semiglobal stabilization can be achieved by linear (high-gain) feedback, provided that the separation between the different gains of the control law is sufficient.
The linear semiglobal design results in a drastic simplification over the Lyapunov global design but an insufficient separation of the gains may cause the region of attraction to shrink.
These results are well documented in the literature (see for instance [10, chap. 6] for a survey and additional references, including [13].

As a complement to backstepping designs, *forwarding* Lyapunov designs have recently been developed [10, 8] for the global stabilization of feedforward systems

$$\begin{array}{rcl} \dot{\xi}_1 &=& \xi_1 h_1(\xi_2,\cdots,\xi_n,z) + \phi_1(\xi_2,\cdots,\xi_n,z) + \psi_1(\xi_2,\cdots,\xi_n, \\ \dot{\xi}_2 &=& \xi_2 h_2(\xi_3,\cdots,\xi_n,z) + \phi_2(\xi_3,\cdots,\xi_n,z) + \psi_2(\xi_3,\cdots,\xi_n, \\ &\vdots& \\ \dot{\xi}_{n-1} &=& \xi_{n-1}h_{n-1}(\xi_n,z) + \phi_{n-1}(\xi_n,z) + \psi_{n-1}(\xi_n,z)u \\ \dot{\xi}_n &=& \xi_n f_n(z) + \phi_n(z) + \psi_n(z)u \\ \dot{z} &=& f(z) + g(z)u \end{array} \tag{1.3}$$

which consist of a core stable system $\dot{z} = f(z)$ augmented by a chain of nonlinear integrators with all the nonlinearities in feedforward form.
The forwarding methodology provides a recursive construction of a Control Lyapunov Function (CLF [9]) which can be employed for the design of a globally stabilizing control law.

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The objective of the present paper is to show that, similarly to the case of systems in feedback form, semiglobal stabilization of feedforward systems is achieved with a (low-gain) linear feedback.
It has been shown ([7]) that a low-gain strategy with a single tunable parameter achieves semiglobal stabilization when the core system is augmented by a chain of integrators.
We show that, when linearly bounded nonlinearities in strict feedforward form are added to the integrators, the same results holds.
In the general case (1.3), the separation between the different gains of the control law must be sufficient because an insufficient separation of the gains may cause the region of attraction to shrink; the single parameter approach has to be replaced by a recursive approach which allows a sufficient separation of the low gains.
In addition to this asymptotic result, we also show how the Lyapunov functions constructed in the recursive global Lyapunov designs can be employed to tune the gains of the linear controller in order to achieve a prescribed region of attraction.

Our results complement the existing semiglobal results for the systems (1.2) and show that semiglobal stabilization by linear feedback can be achieved for any nonlinear system which can be obtained by successive backward and forward augmentations of a core stable subsystem.

Our nested low-gain linear design can be compared to the nested saturation design of Teel [14] for strict-feedforward systems, that is, when the functions h_i in (1.3) are identically zero.
Teel showed that global stabilization results can be obtained in this case if the linear gains are replaced by nonlinear saturations.
The nested saturation design also requires a sufficient separation of the (nonlinear) gains to achieve global stabilization.

The paper is organized as follows.
In Section 2, we describe the two building blocks of recursive semiglobal designs, that is, the semiglobal stabilization by linear feedback of the backward and feedforward augmentation of a stable subsystem by one integrator.
In Section 3, we extend the semiglobal result to a particular class of feedforward systems (1.3) for which the different gains of the linear controller can be tuned with increasing powers of a single parameter ϵ.
We also show through simple examples that such a tuning of the gains does not allow the semiglobal stabilization of general feedforward systems and may lead to vanishing regions of attraction.
The general case is then treated in Section 4, through the recursive application of a semiglobal forwarding result.

2 Backstepping and forwarding an integrator with linear feedback

The two building blocks of recursive designs consist in the backward and feedforward augmentation by one additional integrator of a core system

$$\dot z = f(z) + g(z)u,\ f(0) = 0,\ \ z = (z_1, \ldots, z_p)^T \in I\!R^p \quad (2.4)$$

for which we assume that the equilibrium $z = 0$ of $\dot z = f(z)$ is globally asymptotically stable (GAS) and locally exponentially stable (LES).
By standard converse theorems (see Appendix), there exists a smooth Lyapunov function $U(z)$ which satisfies the following for some constants α_1 and $\alpha_2 > 0$:

(i) $\forall z \in I\!R^p : \alpha_1 \|z\|^2 \le U(z)$

(ii) $\forall z \in I\!R^p : L_f U(z) \le -\alpha_2 \|z\|^2$

A backward augmentation of the system (2.4) by one integrator results in

$$\begin{aligned} \dot z &= f(z) + g(z)\xi \\ \dot \xi &= u \end{aligned} \quad (2.5)$$

The high-gain feedback $u = -k\xi$, k large, enforces a time-scale separation between the convergence of $\xi(t)$ to $\xi = 0$ and the remaining dynamics $\dot z = f(z)$.
An estimate of the region of attraction is obtained by taking the time-derivative of the Lyapunov function $V = U(z) + 1/2\xi^2$, which is

$$\dot V = L_f U(z) + L_g U(z)\xi - k\xi^2 \le -\alpha_2\|z\|^2 + L_g U(z)\xi - k\xi^2$$

Completing the squares, we obtain that $\dot V$ is negative definite in the set Ω where $|L_g U(z)| < 2\sqrt{k\alpha_2}\|z\|$. This means that the region of attraction contains the largest level set of V contained in Ω. Because V is independent of k and Ω tends to the entire state space as $k \to \infty$, the control law $u = -k\xi$ achieves semiglobal stabilization of the equilibrium $(z,\xi) = (0,0)$.

A forward augmentation of the system (2.4) by one integrator results in

$$\begin{aligned} \dot \xi &= z_1 \\ \dot z &= f(z) + g(z)u \end{aligned} \quad (2.6)$$

The linear change of coordinates

$$y = \xi + q^T z,\ \ q = -F^{-T} e_1,\ \ F = Df(0),\ \ e_1 = (1, 0, \ldots, 0)^T \quad (2.7)$$

transforms (2.6) into

$$\begin{aligned} \dot y &= q^T g(z) u + q^T f_1(z) \\ \dot z &= f(z) + g(z)u \end{aligned} \quad (2.8)$$

where $f_1(z) = f(z) - Fz$ only contains the nonlinear part of $f(z)$. If the Jacobian linearization of (2.6) is stabilizable, then $q^T g(0) \neq 0$ and we assume (up to the multiplication of y by a constant) that $q^T g(0) = 1$, that is,

$$q^T g(z) = 1 + q^T g_1(z),\ \ g_1(z) := g(z) - g(0)$$

Instead of the high-gain employed for the system (2.5), we now use the low-gain feedback

$$u = -\epsilon y, \quad (2.9)$$

where a small value for $\epsilon > 0$ enforces a time-scale separation between the convergence of z to a neighborhood of $z = 0$ and the remaining dynamics $\dot y = -\epsilon y + O(\|z\|^2)$.

An estimate of the region of attraction of the equilibrium $(y,z) = (0,0)$ is obtained by using the Lyapunov function [8]

$$V(y,z) = \sqrt{1+y^2} - 1 + \int_0^{U(z)} \gamma(s)ds \qquad (2.10)$$

where $\gamma(s)$ is a positive non integrable function to be determined ($\int_0^{+\infty}\gamma(s)ds = +\infty$ is necessary to ensure that V is radially unbounded). The time-derivative of V along the solutions of (2.8) is

$$\dot V = -\epsilon\frac{y^2}{\sqrt{1+y^2}} + \epsilon\psi(y,z) + \gamma(U(z))L_fU + \frac{y}{\sqrt{1+y^2}}q^Tf_1(z)$$

where the cross-term

$$\psi(y,z) = -\frac{y^2}{\sqrt{1+y^2}}q^Tg_1(z) - \gamma(U(z))L_gU(z)y$$

satisfies $\psi(0,z) = \psi(y,0) = 0$, $D\psi_y(y,0) = 0$, and $D\psi_z(0,z) = 0$.

The terms independent of ϵ satisfy

$$\gamma(U(z))L_fU + \frac{yq^Tf_1(z)}{\sqrt{1+y^2}} \le -\gamma(U(z))\alpha_2\|z\|^2 + |q^Tf_1(z)|$$

Because $q^Tf_1(z)$ is at least quadratic near the origin and U is radially unbounded, we can construct a function $\gamma(s) \ge 1$ such that, for all z,

$$-\gamma(U(z))\alpha_2\|z\|^2 + |q^Tf_1(z)| \le -\alpha_2\|z\|^2$$

With this function γ, we have completed the definition of the Lyapunov function (2.10). Let Ω be the desired region of attraction of $(\xi,z) = (0,0)$. Because V is radially unbounded, we can choose K large enough such that

$$\Omega \subset \mathcal{U}_K = \{(\xi,z)|V(y,z) \le K\}$$

Inside the compact set $\mathcal{U}_K$, there exist constants $k_1 > 0$ and $k_2 > 0$ such that $|\psi(y,z)| \le k_1|y|\|z\|$ and $\frac{1}{\sqrt{1+y^2}} \ge k_2$. The time-derivative $\dot V$ then satisfies in $\mathcal{U}_K$

$$\dot V(y,z) \le -\epsilon k_2y^2 + \epsilon k_1|y|\|z\| - \alpha_2\|z\|^2$$

Completing the squares, we conclude that $\dot V$ is negative definite inside $\mathcal{U}_K$ provided that the constant $\epsilon > 0$ is chosen small enough to satisfy

$$\epsilon < \frac{4k_2\alpha_2}{k_1^2}$$

We conclude that the region of attraction increases to the whole state space as $\epsilon \to 0$, which proves the semiglobal stabilization. Local exponential stability follows from the fact that V and $\dot V$ are quadratic near the origin.

Two conclusions must be retained from the above analysis: on the one hand, an *asymptotic result*, which guarantees that the linear low-gain feedback (2.9) achieves stabilization in any given compact set provided that the gain ϵ is sufficiently low (which had already been proven in [7]). On the other hand, the determination of an upper-bound on e from the Lyapunov function (2.10) in the case when a Lyapunov function is known for the subsystem $\dot z = f(z)$.
In the next two sections,we will extend these conclusions to more complex situations in which several integrators are added to the original system, with nonlinearities in feedforward form.

3 Forwarding a chain of integrators with linear feedback

The next result extends the construction in Section 2 to a forward augmentation of the core subsystem by a chain of integrators.
The analog result for a backward augmentation can be found in [13].

Theorem 1 [7] Consider the system

$$\left\{\begin{array}{lcl} \dot\xi_1 &=& \xi_2 \\ \dot\xi_2 &=& \xi_3 \\ &\vdots& \\ \dot\xi_{n-1} &=& \xi_n \\ \dot\xi_n &=& z_1 \\ \dot z &=& f(z) + g(z)u \end{array}\right. \qquad (3.11)$$

where $\xi \in \mathbb{R}^n$ and $z \in \mathbb{R}^p$. Assume that the Jacobian linearization of (3.11) is stabilizable and that the equilibrium $z = 0$ of $\dot z = f(z)$ is GAS/LES. Let $p(s) = s^n + a_{n-1}s^{n-1} + ... + a_1s + a_0$ be an arbitrary Hurwitz polynomial.

Then the feedback

$$u = -\left(a_0\epsilon^n\xi_1 + a_1\epsilon^{n-1}\xi_2 + \cdots + a_{n-2}\epsilon^2\xi_{n-1} + a_{n-1}\epsilon y_n\right) \qquad (3.12)$$

where $y_n = \xi_n + q^Tz$ as in (2.7), achieves semiglobal stabilization of $(\xi,z) = (0,0)$, that is, the region of attraction of $(\xi,z) = (0,0)$ tends to entire state space as $\epsilon \to 0$.

Proof: Using scaled coordinates as in [7]

$$y_i = \epsilon^{n-i}\xi_i \;\forall\, i \in \{1,\cdots,n-1\}$$

and y_n as in (2.7). Let A be the controller form matrix with characteristic polynomial $p(s)$ and let $P > 0$ be solution of the Lyapunov equation $A^TP + PA = -I$. Then we use the Lyapunov function

$$V(y,z) = \sqrt{1+y^TPy} - 1 + \int_0^{U(z)} \gamma(s)ds \qquad (3.13)$$

where $\gamma(s)$ is a positive non integrable function.
Proceeding as in section 2, we achieve semiglobal stabilization and V can be employed to obtain an upper bound on ϵ such that the system is stabilized in an a priori fixed set.

The above result can be extended to strict feedforward systems under a linear growth assumption for the nonlinearities:

$$\left\{\begin{array}{lcl} \dot\xi_1 &=& \xi_2 + \phi_1(\xi_2,\cdots,\xi_n,z,u) \\ \dot\xi_2 &=& \xi_3 + \phi_2(\xi_3,\cdots,\xi_n,z,u) \\ &\vdots& \\ \dot\xi_{n-1} &=& \xi_n + \phi_{n-1}(\xi_n,z,u) \\ \dot\xi_n &=& z_1 + \phi_n(z,u) \\ \dot z &=& f(z) + g(z)u \end{array}\right. \qquad (3.14)$$

where $\phi(0,...,0) = 0$ and

$$\begin{array}{lcl} \|\phi_i(\xi_{i+1},\cdots,\xi_n,z,u)\| &\le& \gamma_i(\|(\xi_n,z,u)\|)(\|z\|(1+|\xi_{i+1}|) + \|(\xi_{i+2},\cdots,\xi_n)\|) \\ && \text{for } i \in \{1,\cdots,n-1\} \\ \|\phi_n(z,u)\| &\le& \|z\||u|\gamma_n(\|(z,u)\|) \end{array} \qquad (3.15)$$

for some C^1 positive functions γ_i.
Two terms concerning the ξ_i states can be distinguished in this bound: the first one imposes that the dependence of ϕ_i in ξ_{i+1} is at most linear, and must appear in terms proportional to $\|z\|$; the second term of the bound restricts the allowable nonlinearities in $(\xi_{i+2}, ..., \xi_n)$ to linearly bounded ones. Much more freedom is allowed for the nonlinearities in ξ_n, z, and u.

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The proof of Theorem 1 is easily adapted: those nonlinearities only add two kind of terms to the derivative of the Lyapunov function: a cross term $\epsilon\psi(x,y)$ and a term of the form $\epsilon^2 k_3\|y\|^2$, both of which do not disturb the proof.

In contrast, when the growth assumption (3.15) is removed, the control law does not achieve semiglobal stabilization for general strict feedforward systems. This is illustrated by the following examples.

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 + z - (\xi_2 + z)^3\\ \dot{\xi}_2 &= z\\ \dot{z} &= -z + u\end{aligned}$$

Example 1 The system

has the strict feedforward form (3.14) but does not satisfy the growth assumption (3.15) because the nonlinearity is not proportional to z: the first part of the constraint is not satisfied (moreover, the ξ_2 term is cubic). The application of Theorem 1 leads to the change of coordinates $y_2 = \xi_2 + z$, which yields

$$\begin{aligned}\dot{\xi}_1 &= y_2 - y_2^3\\ \dot{y}_2 &= u\end{aligned}$$

The control law of Theorem 1 is $u = \epsilon y_2 = \epsilon^2\xi_1$ and, using the scaled coordinate $y_1 = \epsilon\xi_1$, the closed-loop system is

$$\begin{aligned}\frac{1}{\epsilon}\dot{y}_1 &= y_2 - y_2^3\\ \frac{1}{\epsilon}\dot{y}_2 &= -y_2 - y_1 r\end{aligned} \qquad (3.16)$$

The set

$$E = \{(y_1, y_2) | y_2 - y_1 \geq 2, \;\; y_2 \geq 1\}$$

is invariant, that is, the solutions of (3.16) starting in E remain in E for all $t \geq 0$. This is verified by showing that initial conditions on the boundary of E do not leave E: Defining $\zeta = y_2 - y_1$, we have

$$\dot{y}_2 \mid_{y_2=1} = \epsilon(\zeta - 2) \geq 0 \;\text{ if } \zeta \geq 2$$

and

$$\dot{y}_2 \mid_{y_2=1} = \epsilon(\zeta - 2) \geq 0 \;\text{ if } \zeta \geq 2$$

Because E is invariant and does not contain the equilibrium $(y_1, y_2) = (0,0)$, it has no intersection with the region of attraction, regardless of $\epsilon > 0$. In particular, the region of attraction does not extend along the axis $\xi_1 = 0$ beyond the point $(\xi_1(0), y_2(0)) = (y_1(0), y_2(0)) = (0,2)$.

In the above example, the region of attraction is limited in one direction of the state space as $\epsilon \to 0$. The situation can be even worse. The following example shows that the region of attraction may decrease with ϵ and even vanish as $\epsilon \to 0$.

Example 2 Consider the feedforward system

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 + \xi_3^3\\ \dot{\xi}_2 &= \xi_3\\ \dot{\xi}_3 &= \xi_4 + z\\ \dot{\xi}_4 &= z\\ \dot{z} &= -z + u\end{aligned}$$

We have two perturbations in comparison with the simple 4^{th} order integrator: the z term in equation (3.19), which satisfies the growth constraint and the ξ_3^3 term in equation (3.17), which is not linearly bounded (the second part of the bound is violated).

Placing the poles of the Hurwitz polynomial in -1, we design the control law according to Theorem 1 and obtain:

$$u = -\epsilon^4\xi_1 - 4\epsilon^3\xi_2 - 6\epsilon^2\xi_3 - 4\epsilon(\xi_4 + z)$$

Using the change of coordinates $(y_1, y_2, y_3, y_4) = (\epsilon^{5/2}\xi_1, \epsilon^{3/2}\xi_2, \epsilon^{1/2}\xi_3, \epsilon^{-1/2}\xi_4)$, the closed-loop system is:

$$\begin{aligned}\frac{1}{\epsilon}\dot{y}_1 &= y_2 + y_3^3\\ \frac{1}{\epsilon}\dot{y}_2 &= y_3\\ \frac{1}{\epsilon}\dot{y}_3 &= y_4\\ \frac{1}{\epsilon}\dot{y}_4 &= -y_1 - 4y_2 - 6y_3 - 4y_4\end{aligned}$$

Simulations show that a solution of this system with initial condition $(y_1, y_2, y_3, y_4) = (0,0,0,30)$ grows unbounded.

In the original coordinates, it means that the closed loop system with initial condition $(\xi_1, \xi_2, \xi_3, \xi_4, z) = (0,0,0,30\sqrt{\epsilon}, 0)$ is unbounded. It shows that the region of attraction shrinks in the ξ_4 direction when $\epsilon \to 0$.

This could be interpreted as follows: while $\xi_4 + z$ converges to the origin at a rate of order ϵ, there is a peak of order $1/\epsilon$ in ξ_3 and of order $1/2\epsilon$ in ξ_2. The peak of the ξ_1 variable will be of order $1/4\epsilon$ because of the ξ_3^3 term.

Without this cubic term, the variable would have peaked at an order of $1/3\epsilon$, which would have not destabilized the system and would have allowed the region of attraction of the origin to extend while we decrease ϵ.

In the next section we will see that the vanishing region of attraction in Example 2 is due to an insufficient separation of the gains ϵ_i as $\epsilon \to 0$. The class of strict-feedforward systems (3.14) with a linear growth assumption (3.15) thus covers a special situation in which several integrators can be forwarded in one design step, the different gains of the controller being tuned with increasing powers of a single parameter ϵ. To avoid vanishing regions of attractions in the general case (1.3), it will be necessary to proceed in n different steps, a new tuning parameter ϵ_i being defined at each step.

4 Recursive semiglobal stabilization of feedforward systems

To achieve the semiglobal stabilization of feedforward systems (1.3) in a recursive way, we will now extend the result of Section 2 in two directions —compare with (2.6): first, to allow a recursive application of the result, it is necessary to start from a core system which is not necessarily GAS: we will only assume that the equilibrium $z = 0$ of $\dot{z} = f(z)$ is locally exponentially stable (LES) and has a region of attraction $\mathcal{A}$ which contains the compact set $\Omega_z \subset \mathbb{R}^p$ to be included in the prescribed region of attraction.
This is only a minor modification with respect to Section 2 because, as shown in Appendix, converse theorems guarantee the existence of a Lyapunov function $U(z)$ which satisfies the following for some constants $\alpha_1, \alpha_2 > 0$ ($\partial\mathcal{A}$ denotes the boundary of $\mathcal{A}$):

(i) $\alpha_1\|z\|^2 \leq U(z)$ and $\lim_{z\to\partial\mathcal{A}} U(z) = +\infty$

(ii) $L_f U \leq -\alpha_2\|z\|^2$

A second extension with respect to Section 2 is that we consider the more general forward augmentation

$$\begin{aligned} \dot{\xi} &= h_1(z) + \xi h_2(z) + h_3(z)u, \quad h_1(0) = 0 \\ \dot{z} &= f(z) + g(z)u \end{aligned} \tag{4.22}$$

where $h_1(0) = 0$ and $h_2(z)$ is at least quadratic near the origin, that is, $h_2(0) = 0$ and $Dh_2(0) = 0$.

The linear change of coordinates

$$y = \xi + q^T z, \quad q^T = -Dh_1(0)F^{-1}$$

transforms the first equation of (4.22) into

$$\dot{y} = (h_3(0) + q^T g(0))u + h.o.t.$$

where *h.o.t.* denotes higher-order terms. If the Jacobian linearization of (4.22) is stabilizable, then $h_3(0) + q^T g(0) \neq 0$. Up to the multiplication of y by a constant, we assume without loss of generality that $h_3(0) + q^T g(0) = 1$. We then rewrite the system (4.22) as

$$\begin{aligned} \dot{y} &= \bar{h}_1(z) + yh_2(z) + (1 + h_3^T z + h_4(z))u, \\ \dot{z} &= f(z) + g(z)u \end{aligned} \tag{4.23}$$

where

$$\bar{h}_1(z) := h_1(z) - Dh_1(0)z - h_2(z)q^T z + q^T(f(z) - Fz), \quad \bar{h}_1(0) = 0, D\bar{h}_1(0) = 0$$

and

$$h_3^T z + h_4(z) := h_3(z) - h_3(0) + q^T(g(z) - g(0)) \quad h_4(0) = 0, Dh_4(0) = 0$$

Then we have the following result.

Theorem 2 Let $\Omega = \Omega_\xi \times \Omega_z \subset \mathbb{R} \times \mathcal{A}$ be any compact set. Then there exists $\bar{\epsilon} > 0$ such that, for all $0 < \epsilon \leq \bar{\epsilon}$, the equilibrium $(\xi, z) = (0,0)$ of (4.22) is locally exponentially stable with the control law $u = -\epsilon y$ and the region of attraction includes Ω.

Proof: We define the Lyapunov function

$$V(y, z) = \int_0^{U(z)} \gamma(s)ds + ln(1 + y^2)$$

where $\gamma(s) \geq 1$ is a continuous function so that $V(y,z)$ is positive definite function and radially unbounded in $\mathbb{R} \times \mathcal{A}$. Its time-derivative is

$$\dot{V}(y, z) = -\epsilon\frac{2y^2}{1 + y^2} + \epsilon\psi(z, y) + \gamma(U)L_f U + \frac{2y}{1 + y^2}(\bar{h}_1(z) + yh_2(z) + \epsilon y h_4(z)) \tag{4.24}$$

where the cross-term $\psi(y,z)$ is

$$\psi(y, z) = \frac{2y^2}{1 + y^2}h_3^T z - \gamma(U)L_g U(z)y$$

Because the functions $L_f U(z), \bar{h}_1(z), h_2(z)$, and $h_4(z)$ are all at least quadratic near the origin, and U is radially unbounded in $\mathcal{A}$, we can choose $\gamma(s)$ such that

$$\gamma(U)L_f U + \frac{2y}{1 + y^2}(\bar{h}_1(z) + yh_2(z) + \epsilon y h_4(z)) \leq -\alpha\|z\|^2 \quad \forall z \in \mathcal{A}$$

We obtain in this way

$$\dot{V} \leq -\epsilon\frac{2y^2}{1 + y^2} + \epsilon\psi(z, y) - \alpha\|z\|^2$$

The choice of γ completes the definition of V. Because U is radially unbounded in $\mathcal{A}$, there exists a constant $K > 0$ large enough such that $\|(\xi,z)\| \in \Omega \Rightarrow (\xi, z) \in \mathcal{U}_K = \{(\xi, z) | V(y,z) \leq K\}$.
There exist two constants $k_1 > 0$ and $k_2 > 0$ such that $|\psi(y, z)| \leq k_1|y|\,\|z\|$ and $\frac{2}{1+y^2} \geq k_2$ inside the compact set $\mathcal{U}_K$. Completing the squares, we conclude that $\dot{V}$ is negative definite inside $\mathcal{U}_K$ provided that the constant $\epsilon > 0$ is chosen small enough to satisfy

$$\epsilon < \frac{4\alpha k_2}{k_1^2}$$

Local exponential stability follows from the fact that V and $\dot{V}$ are quadratic near the origin.
A recursive application of Theorem 2 yields the following conclusion. (The corresponding result for systems in the feedback form (1.2) can be found in [13]).

Theorem 3 Consider the feedforward system

$$\left\{\begin{aligned} \dot{\xi}_1 &= \xi_1 h_1(\xi_2, \cdots, \xi_n, z) + \phi_1(\xi_2, \cdots, \xi_n, z) + \psi_1(\xi_2, \cdots, \xi_n, z)u \\ \dot{\xi}_2 &= \xi_2 h_2(\xi_3, \cdots, \xi_n, z) + \phi_2(\xi_3, \cdots, \xi_n, z) + \psi_2(\xi_3, \cdots, \xi_n, z)u \\ &\vdots \\ \dot{\xi}_{n-1} &= \xi_{n-1}h_{n-1}(\xi_n, z) + \phi_{n-1}(\xi_n, z) + \psi_{n-1}(\xi_n, z)u \\ \dot{\xi}_n &= \xi_n h_n(z) + \phi_n(z) + \psi_n(z)u \\ \dot{z} &= f(z) + g(z)u \end{aligned}\right. \tag{4.25}$$

where for each i, $h_i(0) = 0$ and $Dh_i(0) = 0$, that is, h_i is at least quadratic near theorigin.
Assume that the Jacobian linearization of (4.25) is stabilizableand that the equilibrium $z = 0$ of $\dot{z} = f(z)$ is locally exponentially stablewith a region of attraction $\mathcal{A} \subset \mathbb{R}^p$. Let $\Omega = \Omega_\xi \times \Omega_z \subset \mathbb{R}^n \times \mathcal{A}$ be an arbitrary compact set. Then there exists constants $\bar{\epsilon}_i > 0$ such that, for any $0 < \epsilon_i \leq \bar{\epsilon}_i$, the equilibrium $(\xi, z) = (0,0)$ of (4.25) is locally exponentially stable with a linear control law of the form

$$u = -\sum_{i=1}^{n} \epsilon_i y_i, \quad y_i = \sum_{k=i}^{n} \alpha_{ik}\xi_k + q_i^T z$$

and its region of attraction contains Ω. (The coefficients α_{ik} and the vectors q_i depend on the parameters ϵ_k, $k > i$.)

5 Conclusion

This paper has addressed the semiglobal stabilization by linear feedback ofa large class of feedforward nonlinear systems.
We have used the Lyapunov construction proposed in [8] for the global stabilization of feedforward systems to estimate the gains of the linear controller needed to achieve a prescribed region of attraction.
We have identified particular situations in which the tuning of the gains can be achieved with increasing powers of a single parameter.
We have shown that such a simple tuning may cause vanishing regions of attraction for general feedforward systems.
In this case, the tuning of n independent parameters must be achieved in a recursive way to guarantee arbitrarily large regions of attractions.

A Converse theorems

The converse theorem used in Section 2 can be deduced from standard converse theorems, see for instance [5]. A simple proof, given in [10, Lemma B.1.], defines the Lyapunov function $U(z)$ as the line integral

$$U(z) = \int_0^\infty \|\bar{z}(s)\|^2 ds \tag{A.26}$$

where $z(s)$ is a solution of the scaled system

$$\dot{\bar{z}} = \frac{1}{1+\|f(\bar{z})\|^2} f(\bar{z}) := F(\bar{z}), \quad \bar{z}(0) = z$$

The time-derivative of U along the solutions of $\dot{z} = f(z)$ yields

$$\dot{U} = L_f U(z) = -(1 + \|f(z)\|)\|z\|^2 \leq -\|z\|^2$$

On the other hand, thanks to the linear growth of F

$$\|F(z)\| \leq L\|z\|, \quad L = \max\left\{1, \sup_{\|z\|\leq 1} \left\|\frac{\partial f}{\partial z}(z)\right\|\right\}$$

we can use

$$\frac{d}{ds}\|\bar{z}\|^2 = 2\bar{z}^T F(\bar{z}) \geq -2L\|\bar{z}\|^2$$

to obtain $\|z(s)\|^2 \geq e^{-2Ls}\|z\|^2$ and prove

$$U(z) \geq \int_0^\infty e^{-2Ls}\|z\|^2 \, ds = \alpha_1 \|z\|^2$$

The same definition (A.26) can be used for the converse theorem in Section 4, that is, when the region of $z = 0$ is an open set $\mathcal{A} \subset \mathbb{R}^n$. We only need to establish the additional property

$$\lim_{z \to \partial\mathcal{A}} U(z) = +\infty \tag{A.27}$$

Choose $\delta > 0$ such that the ball $B(0,\delta)$ of radius $\delta > 0$ is contained in $\mathcal{A}$.
For each $z \in \mathcal{A}/B(0, \delta)$; define $T(z) \geq 0$ as the time needed for the solution $z(s)$ to reach the ball $B(0,\delta)$, that is,

$$T(z) = \inf_{t\geq 0}\{t : \|\bar{z}(t)\| \leq \delta\}$$

Then we have for each $z \in \mathcal{A}/B(0, \delta)$:

$$U(z) \geq \int_0^{T(z)} \|\bar{z}(s)\|^2 ds \geq T(z)\delta$$

By standard stability theorems (see for instance [5, Theorem 33.2]), $T(z) \to \infty$ as $z \to \partial\mathcal{A}$, which proves the property (A.27).

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